Auction Algorithms for Market Equilibrium under Weak Gross Substitute Demands

Jugal Garg, Edin Husić, and László Végh

Fisher market

- Set of m divisible goods G.
- Set of n agents A, each with budget b_i.

Goal: find prices such that market clears

supply = demand

- *Supply*: wlog there is one unit of each good $j \in G$,

$$\sum_{i \in A} e_{ij} = 1 \, .$$

Demand and equilibrium

A ***demand*** is a function $D_i : \mathbb{R}_+^{G+1} \to \mathbb{R}_+^G$;

$D_i(p, b_i)$ is the *preferred* bundle of an agent i at prices p and budget b_i.

Bundle = vector of goods.
Preferred or optimal or demanded.

Definition **[Market equilibrium]**:

We say that the prices $p \in \mathbb{R}_+^G$ and bundles $x_i \in \mathbb{R}_+^G$ form a ***market equilibrium*** if

- $x_i = D_i(p, b_i)$, and
- $\sum_{i=1}^{n} x_{ij} \leq 1$ with equality whenever $p_j > 0$, for all $j \in G$.

Tâtonnement

- Informally:
    1. Start with arbitrary prices $p \in \mathbb{R}_{+}^{G}$.
    2. Look at the excess demand (demand - supply) and "fix" the price of a single good:
       change the price of good j until the demand = supply on good j.
    3. Repeat.
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When does it converge?

Intuition. Suppose one price raises. Then we expect that the demand for that good falls.

Purchasing power is then diverted to the other goods; it is reasonable to assume that demand for other goods increases.

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Definition **[Weak gross substitutes]**:

Consider price vectors $p, q \in \mathbb{R}_+^G$ such that $p \leq q$ (pointwise). Demand D_i of agent i satisfies *weak gross substitutes* property if

$$\text{for } x_i = D_i(p, b_i) \text{ and } y_i = D_i(q, b_i) \text{ it holds } y_{ij} \geq x_{ij} \text{ whenever } p_j = q_j.$$

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- Introduced by Arrow, Block and Hurwicz in 1958, 1959.
- They showed that *tâttonement* converges to an equilibrium if *aggregate* demand satisfies **GS**.

$$\sum_{i \in A} D_i(p, b_i)$$

Examples of WGS utilities/demands

Assume agent i is equipped with a concave utility function $u_i : \mathbb{R}^G_+ \to \mathbb{R}_+$, then we have

$$D_i(p, b_i) := D^{u_i}(p, b_i) := \arg\max\{u_i(x_i) : p^\top x_i \le b_i, x \ge 0\}$$

Linear (additive) utility

$$u(x) = v^\top x \quad \text{for } v \in \mathbb{R}^G_+$$

$$D^u(p, b) = \arg\max\{v^\top x : p^\top x \le b\}$$

$$= \{x \in \mathbb{R}^G : \frac{v_j}{p_j} \text{ is maximum, and } p^\top x = b\}$$

$$MBB = \max_{j \in G} \frac{v_j}{p_j}$$

Constant elasticity of substitution

$$u(x) = \left(\sum_j \beta_j^{\frac{1}{\sigma}} x_j^{\frac{\sigma - 1}{\sigma}} \right)^{\frac{\sigma}{\sigma - 1}}$$

$$D^u(p, b) = x \text{ where}$$

$$x_j = \frac{\beta_j p_j^{-\sigma} b}{\sum_k \beta_k p_k^{1-\sigma}}.$$

The demand (utility) is GS iff $\sigma \ge 1$.

The Cobb-Douglas utility

$$u(x) = \prod_j x_j^{\alpha_j} \text{ s.t. } \sum_j \alpha_j = 1, \alpha_j \ge 0$$

$$D^u(p, b) = x \text{ where}$$

$$x_j = b\alpha_j / p_j.$$

Complexity of finding an equilibrium?

For WGS utilities:

- The first polytime algorithm. Codenotti, Pemmaraju, and Varadarajan [2005]
- A simple ascending price algorithm. Bei, Garg, and Hoefer [2019]
- A discrete variant of tâttonement converges to an approximate equilibrium. Codenotti, McCune, and Varadarajan [2005]
- A lot more…

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Outside of WGS:

- In general, hopeless. Finding equilibria when utilities are "just outside" gross substitutability is PPAD-complete. Chen, Paparas, Yannakakis [2013]
- Polynomial time algorithms for particular classes of utilities.

Auction Algorithms

- A subclass of *tâttonement* where prices only go up.
- Under simple “ground rules” the agents outbid each other and converge to an approximate equilibrium.
- Does not require a central authority.
- Robust: small changes allow for various extensions and generalisations.

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- Auction algorithms for *assignment* and *transportation* problems.
  Bertsekas [1981, 1990].
- A long history of auction algorithms for markets with *indivisible goods.*
  Kelso and Crawford [1982], Demange, Gale and Sotomayor [1986].
- Auction algorithm for market equilibrium in exchange market with *linear utilities*.
  Garg and Kapoor [2004]
- Extended to *restricted* subclasses of WGS utilities.
  Garg, Kapoor and Vazirani [2004], Garg and Kapoor [2007].
- **Open:** Design auction algorithm for whole WGS?

Auction algorithm for finding approximate market equilibria in Fisher markets when agents have WGS demands.

Algorithm overview and “ground rules”

- We maintain *market prices* p;
  - Increases only by factor $(1+\epsilon)$.
  - A part $l_j > 0$ of each good is sold at p_j, and the rest is sold at $(1+\epsilon)p_j$. (All goods are fully sold.)

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- Agent i maintains *individual prices* $p^{(i)}$ such that $p_j \le p_j^{(i)} \le (1+\epsilon)p_j$.
  1. Throughout, i owns a bundle c_i such that $c_i \le x_i = D_i(p^{(i)}, b_i)$.
  2. If $p_j^{(i)} < (1+\epsilon)p_j$ agent i pays p_j for the amount c_{ij}.
  3. Otherwise, ($p_j^{(i)} = (1+\epsilon)p_j$) agent i pays $(1+\epsilon)p_j$ for c_{ij}.

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Local for agent

Consider the agents one-by-one. If agent has surplus, she will try use it to get more goods by outbidding.

Main ingredient

- Throughout, i owns a bundle c_i such that $c_i \leq x_i = D_i(p^{(i)}, b_i)$.

FindNewPrices$(p^{(i)}, c^{(i)}, b_i)$ delivers new prices $\tilde{p}$:

A. $y \geq c_i$ for $y = D_i(\tilde{p}, b_i)$, and

B. $p^{(i)} \leq \tilde{p} \leq (1+\epsilon)p$, where $\tilde{p}_j = (1+\epsilon)p_j$ whenever $y_j > (1+\epsilon)c_{ij}$.

Agent willing to outbid when she wants more

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Can be implemented in various ways:

- Linear (additive) utilities: direct algorithm and/or convex programming approach.
- CES and Cobb-Douglas: solve a convex program.
- WGS demands with bounded elasticity: adjustment procedure.

Algorithm: initialisation

- Initialisation: pick low enough prices so some agents demands all of the goods.
- Algorithm is partitioned into **iterations**;
  each iteration finishes when price of a good increases from p_j to $(1+\epsilon)p_j$ (when $l_j = 0$).
- An iteration is partitioned into **steps:**

  If agent has surplus, she will try use it to get more goods by outbidding.

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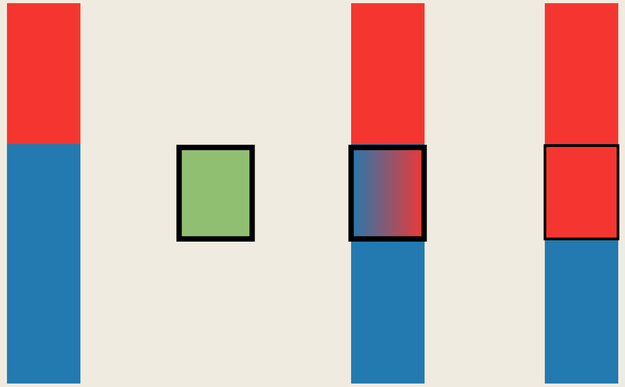
If agent has surplus, she will try use it to get more goods by outbidding.

Outbid: pay higher price $p_j(1+\epsilon)$ to take a part of j currently sold at p_j.

Goods change the owner only through the outbid.



Algorithm: step of agent i

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- $\forall j \in G$ do the corresponding update:
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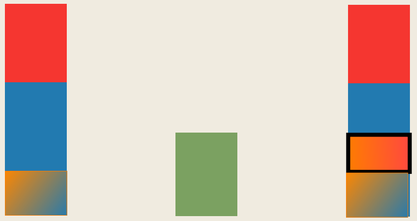

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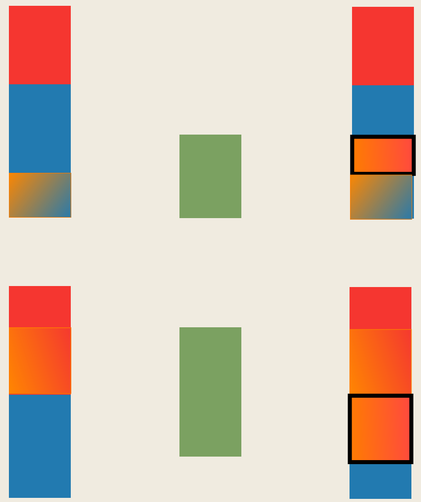

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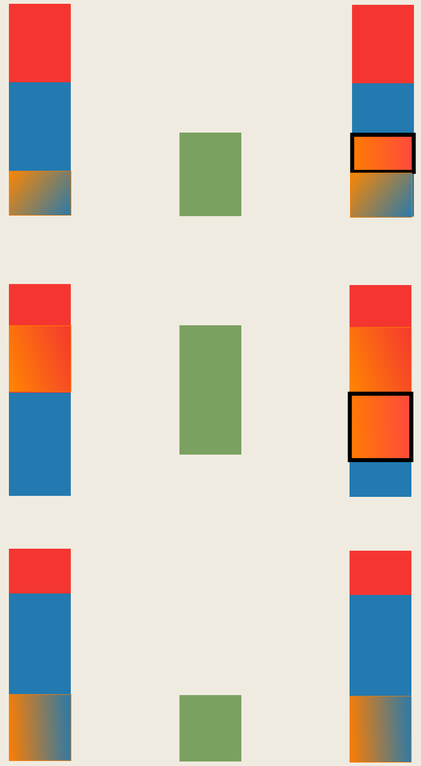

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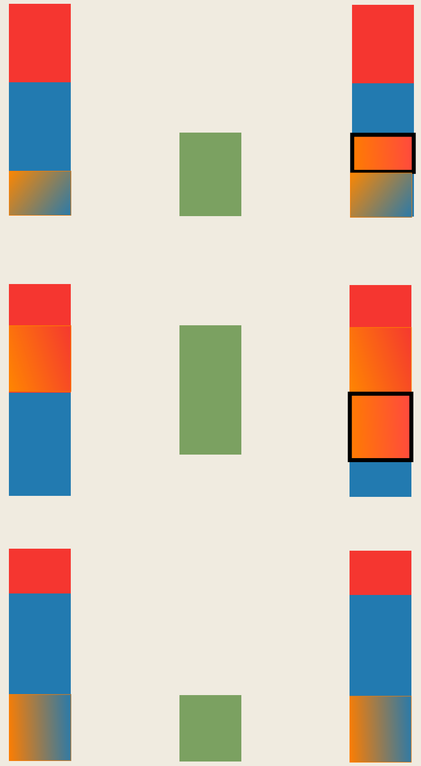

Agent either gets y or increases the price of a good!

Running time

Iterations

In each iteration a price increases by factor $(1+\epsilon)$. Price of a good is at most $\sum_{i \in A} b_i$.

At most $O\left(\frac{m}{\epsilon} \log \frac{\sum_i b_i}{p_{\min}}\right)$ prices increases/iterations.

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Steps

Consider n consecutive steps – **a round**.

Assume the price did not increase, agent i in her turn acquired all she wanted through outbid.

As outbid pays $(1+\epsilon)$ more, the amount of money spent on the goods increases.

Equivalently, the total surplus decreases by factor $(1+\epsilon)$ in each round.

Eventually, we either finish or increase the price.

Recap, comments and applications

- Auction algorithm for finding approximate market equilibria in exchange markets when agents have WGS demands.
- Generalizes to more general **exchange markets**.
- Generalizes to **spending-restricted** market equilibria, recently proposed as a relaxation of the discrete Nash Social Welfare problem.
- Can be extended to markets where WGS is satisfied only approximately.

Thank you!